

QCD, monopoles on the lattice and gauge invariance.

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Abstract. The number and the location of the monopoles observed on the lattice in QCD configurations happens to depend strongly on the choice of the gauge used to expose them, in contrast to the physical expectation that monopoles be gauge invariant objects. It is proved by use of the non abelian Bianchi identities (NABI) that monopoles are indeed gauge invariant, but the method used to detect them depends, in a controllable way, on the choice of the abelian projection. Numerical checks are presented.

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INTRODUCTION

Monopoles play an important role in non abelian gauge theories. They can condense in the vacuum and produce dual superconductivity, which is a good candidate to be the mechanism of color confinement[1][2][3]. The prototype monopole configuration is the soliton solution of Ref.'s [4, 5] in the Higgs-broken phase of an $SU(2)$ gauge theory coupled to a scalar field in the adjoint representation (Georgi-Glashow model). The property which characterizes it as a monopole is a non trivial homotopy π_2 : it realizes a non trivial mapping of the sphere S_2 at spatial infinity onto $SU(2)/U(1)$. This property is invariant under continuous gauge transformations.

A monopole in non-abelian gauge theory is an intrinsically abelian object: the magnetic monopole term in a multipole expansion obeys abelian equations and identifies a $U(1)$ subgroup of the gauge group, modulo an arbitrary global rotation \times an arbitrary gauge transformation $U(\vec{r})$ which tends to the identity as $r \rightarrow \infty$ [6]. In the soliton solution of Ref.[4, 5] this $U(1)$ are the rotations around the color direction of the vacuum expectation of the Higgs field. The corresponding abelian field tensor is known as 't Hooft tensor [3]. In QCD there is no Higgs field and apparently no privileged residual $U(1)$ symmetry. It was proposed in Ref.[3] that any operator O in the adjoint representation can act as an effective Higgs field, the physics being for some reason independent of that choice. Any choice of O identifies an "effective" unitary representation or Abelian Projection.

The recipe to detect monopoles in $U(1)$ lattice configurations[7] relies on the observation that any excess over 2π of the magnetic flux through a plaquette

is a Dirac string crossing it. If the net number of Dirac strings crossing the border of an elementary cube is non zero, there is a monopole inside the cube. In $U(1)$ gauge theory the magnetic flux is gauge invariant, and hence this procedure is gauge invariant and unambiguous. In non-abelian theories one first fixes a gauge (abelian projection), and then plays the same game as in $U(1)$ on the residual $U(1)$ subgroup. The result is in this case strongly dependent on the choice of the gauge, so that the existence or non-existence of a monopole in a site is a gauge dependent feature, and this is physically unacceptable. Either monopoles can be created or destroyed by a gauge transformation, and therefore are unphysical, or the statement of Ref.[3] that all abelian projections are physically equivalent is not correct.

We will solve this problem by use of the non-abelian Bianchi identities (NABI).

THE NON-ABELIAN BIANCHI IDENTITIES

The $U(1)$ (abelian) Bianchi identities $\partial_\mu F_{\mu\nu}^* = 0$, or $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \wedge \vec{E} + \partial_t \vec{B} = 0$ imply zero magnetic current. A violation has the form $\partial_\mu F_{\mu\nu}^* = j_\nu$. The magnetic current j_ν is conserved $\partial_\nu j_\nu = 0$, because of the antisymmetry of $F_{\mu\nu}^*$.

The non-abelian Bianchi identities (NABI) read

$$D_\mu G_{\mu\nu}^* = J_\nu \quad (1)$$

a gauge covariant equation. It implies[8]

$$D_\nu J_\nu = 0 \quad (2)$$

The four components of the magnetic current J_μ commute with each other[9].

¹ presentd the talk

To extract the physical (gauge-invariant) information contained in the *NABI* Eq.(1), we can diagonalize them by a gauge transformation and project on a complete set of diagonal matrices, ϕ_0^a ($a = 1, \dots, r$), r rank of the gauge group.

A convenient choice for ϕ_0^a are the fundamental weights : there is one of them for each simple root of the Lie algebra $\vec{\alpha}^a$.

$$[\phi_0^a, H_i] = 0, [\phi_0^a, E_{\pm\vec{\alpha}}] = \pm(\vec{c}^a \vec{\alpha}) E_{\pm\vec{\alpha}}, (\vec{c}^a \vec{\alpha}^b) = \delta_{ab}.$$

If we call ϕ_I^a the adjoint matrix which is equal to ϕ_0^a in the representation in which J_μ is diagonal the projection gives

$$Tr(\phi_I^a D_\mu G_{\mu\nu}^*) = Tr(\phi_I^a J_\nu) \equiv j_\nu^a(x, I) \quad (3)$$

We can also project on the adjoint matrix ϕ_V^a which is equal to $V(x)\phi_0^a V^\dagger(x)$, $[V(x)$ any gauge transformation], in the gauge where J_μ is diagonal. ϕ_V^a coincides with ϕ_I^a in the special case $V = I$.

$$Tr(\phi_V^a D_\mu G_{\mu\nu}^*) = Tr(\phi_V^a J_\nu) \equiv j_\nu^a(x, V) \quad (4)$$

Let $F_{\mu\nu}^a(x, V)$ be the 't Hooft tensor, i.e. the abelian field strength $\partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ in the gauge (abelian projection) in which ϕ_V^a is diagonal.

In Ref.[8] we proved the following

THEOREM For a generic compact gauge group and for any $V(x)$ Eq.(4) is equivalent to

$$\partial_\mu F_{\mu\nu}^{a*}(x, V) = j_\nu^a(x, V) \quad (5)$$

The abelian Bianchi identities in a generic abelian projection are a consequence of the *NABI*.

THE SOLITON MONOPOLE REVISITED.

We now check our theorem Eq.(5) on the soliton configuration of Ref.[4, 5]. The lagrangean is

$$L = -\frac{1}{4} \vec{G}_{\mu\nu} \vec{G}_{\mu\nu} + (D_\mu \phi)^\dagger (D_\mu \phi) - V(\phi^2)$$

The hedgehog gauge is defined as that in which the vev of the Higgs field $\vec{\phi}(\vec{r}) = H(r)\hat{r}$, $H(r)_{r \rightarrow \infty} \rightarrow v$. In this gauge the solution is

$$\vec{A}_0 = 0 \quad A_i^a = -\epsilon_{iak} \frac{r^k}{gr^2} [1 - K(gvr)]$$

$$[1 - K(x)]_{x \rightarrow 0} \propto x^2, \quad K(x)_{x \rightarrow \infty} \approx \exp(-x)$$

This gauge is nothing but the Landau Gauge. Indeed $\partial_\mu A_\mu = \partial_i A_i = 0$.

The 't Hooft tensor is $F_{ij} = \partial_i A_j^3 - \partial_j A_i^3$ and the abelian magnetic field $b_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$. Direct computation gives

$$\vec{b} \approx_{r \rightarrow \infty} \frac{2\hat{r}}{gr^2} \cos(\theta) \quad (6)$$

$$Q_m = r^2 \int d\Omega \hat{r} \cdot \vec{b} = \frac{2}{g} \int d\Omega \cos(\theta) = 0 \quad (7)$$

No magnetic charge !

In the unitary gauge: $\phi_i \sigma_i = v \sigma_3$. The solution is static and hence $J_i = 0$ ($i = 1, \dots, 3$). For the charge density

$$J_0 = D_i B_i = \frac{2\pi}{g} \delta^3(\vec{r}) \sigma_3 \quad (8)$$

By direct calculation [8] it is found that in this gauge the Maximal Abelian gauge condition satisfied

$$\partial_\mu A_\mu^\pm \pm ig [A_\mu^3, A_\mu^\pm] = 0$$

In $SU(2)$ there is only one fundamental weight $\phi_0 = \frac{\sigma_3}{2}$ (Rank of $SU(2) = 1$).

Our theorem Eq.(5) gives

$$Tr(\phi_0 \vec{D} \cdot \vec{B}) = \vec{\nabla} \cdot \vec{b} = Tr(\phi_0 J_0) = \frac{2\pi}{g} \delta^3(\vec{r}) \quad (9)$$

or

$$Q_m = \frac{1}{g} \quad (10)$$

We can compute the magnetic charge in gauges interpolating between maximal abelian and Landau acting on the maximal abelian with

$$U_a(\theta, \phi) = \exp(i\phi \frac{\sigma_3}{2}) \exp(i\theta \mathbf{a} \frac{\sigma_2}{2}) \exp(-i\phi \frac{\sigma_3}{2}) \quad (11)$$

For $\mathbf{a} = 0$ $U_0(\theta, \phi) = 1$, and one stays in the maximal abelian, for $\mathbf{a} = 1$ $U_1(\theta, \phi)$ transforms to Landau gauge.

Direct computation gives [12]

$$\frac{Q_m(\mathbf{a})}{Q_m(0)} = \frac{1 + \cos(\mathbf{a}\pi)}{2} \leq 1 \quad (12)$$

The magnetic charge measured by the flux at infinity is maximum in the maximal abelian and decreases to zero approaching the Landau gauge.

By use of the analysis of Ref.[6] one can show that the above analysis holds not only for the soliton monopole, but for a generic configuration, which can be viewed as a point-like magnetic charge producing the monopole field at large distances plus a background of zero total magnetic charge responsible for higher multipoles[8].

What we have discussed for the gauge group $SU(2)$ can be extended to any gauge theory, with and without coupling to quark fields [8] [10]. In the following we shall therefore omit the index a in the currents and in the fields since in $SU(2)$ there is only one fundamental weight.

CONFINEMENT

An order parameter for dual superconductivity is the vev of a gauge invariant operator $O(x)$ carrying magnetic

charge: if $\langle O \rangle \neq 0$ the magnetic $U(1)$ is Higgs broken and there is superconductivity (confinement), whilst in the deconfined phase $\langle O \rangle = 0$ [11].

The magnetic charge density operator in the maximal-abelian gauge corresponds to $V(x) = I$ in Eq.(5), i.e. to the current defined by Eq. (3), $j_0(x, I) = \text{Tr}[\phi_I J_0(x)]$; the magnetic charge is $Q_I = \int d^3x j_0(x, I)$. Suppose that $\langle O(x) \rangle \neq 0$ and $[Q, O(x)] = mO(x)$ $m \neq 0$ i.e. that the residual $U(1)$ in the maximal abelian gauge is Higgs-broken. As an element of the Lie algebra $V(x)\phi_0 V^\dagger(x) = C(V, x)\phi_0 + \sum_{\vec{\alpha}} C_{\vec{\alpha}}(V, x)E_{\vec{\alpha}}$ with $\vec{\alpha}$ the roots of the algebra. Computing $j_0(V, x) = \text{Tr}[V(x)\phi_0 V^\dagger(x)J_0(x)]$ in the representation where J_0 is diagonal one gets $j_0(x, V) = C(V, x)j_0(x, I)$ and

$$[Q_V, O(x)] = C(V, x)O(x) \quad (13)$$

Since $C(V, x)$ is generically non zero $O(x)$ is magnetically charged also with respect to the $U(1)$ selected by the Higgs field $V(x)\phi_0 V^\dagger(x)$, and also the corresponding $U(1)$ is Higgs broken. Dual superconductivity is abelian projection invariant.

NUMERICAL CHECK ON THE LATTICE

To check the idea that monopoles are gauge invariant objects, and that it is the detection method of Ref.[7] which is abelian projection dependent, we produce by standard methods an ensemble of $SU(2)$ configurations in the maximal abelian gauge and we look for monopoles [12]. We identify the elementary cubes containing a monopole, and for the sake of simplicity we only consider the cubes with one single Dirac string crossing the border: in principle there could be many of them, corresponding e.g. to strings going through the cube, but we expect them to become irrelevant in the continuum limit. We then assume that the monopole is at the centre of the cube and that the direction of the string is perpendicular to the face crossed, which we assume as 3-rd axis. We then perform the gauge transformation Eq.(11) depending on the parameter α and we measure the magnetic charge by the method of Ref.[7] The result is shown in Fig.(1), for different values of the coupling $\beta \equiv \frac{4}{g^2}$ and compared to the prediction Eq.(12).

The agreement is very good if we consider the systematic errors due to discretization [12], which are, among others, responsible for the slight β dependence observed.

This is a clear demonstration that monopole existence is a gauge invariant fact: observing it in different abelian projections by the recipe of Ref.[7] gives a probability of detection which is always less than 1, and is equal to 1 in the maximal abelian gauge. It is not correct to speak of monopoles in different abelian projections as

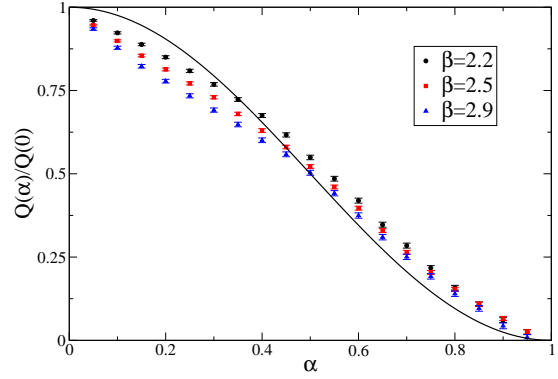


FIGURE 1. Results and theoretical prediction (black line) for the ratio defined in Eq.(12).

if they were different objects. In particular it is not true that no monopoles exist in the Landau gauge: the correct statement is that the monopoles are gauge independent, but they escape detection with the recipe of Ref.[7] in the Landau gauge.

In conclusion, by use of the non abelian Bianchi identities, we have shown that the existence of a monopole in a gauge field configuration is a gauge invariant concept. In a magnetically charged configuration a privileged direction in color space exists, that of the magnetic monopole term in the multipole expansion of the field, which is selected by the maximal abelian gauge.

Confinement is a projection independent property.

REFERENCES

1. G. 't Hooft, in *High Energy Physics*, EPS International Conference, Palermo 1975, A. Zichichi ed.
2. S. Mandelstam, *Phys. Rep.* **23C**, 245 (1976).
3. G. 't Hooft, *Nucl. Phys. B* **190**, 455 (1981).
4. G. 't Hooft, *Nucl. Phys. B* **79**, 276 (1974).
5. A. M. Polyakov, *JETP Lett.* **20**, 194 (1974).
6. S. Coleman, *The Magnetic Monopole Fifty Years Later*, Lectures given at the International School of Subnuclear Physics, Erice, Italy (1981).
7. T. A. DeGrand, D. Toussaint, *Phys. Rev. D* **22**, 2478 (1980). Erice, Italy (1981).
8. C. Bonati, A. Di Giacomo, L. Lepori and F. Pucci, *Phys.Rev.D* **81** 085022 (2010).
9. S. R. Coleman, J. Mandula, *Phys. Rev.* **159**, 1251(1967).
10. A. Di Giacomo, L. Lepori and F. Pucci, *JHEP* **0810**, 096 (2008).
11. A. Di Giacomo, *Acta Phys.Polon.B* **25**:215-226,1994.
12. C. Bonati, M. D'Elia, A. Di Giacomo, Detecting monopoles on the lattice. e-Print: arXiv:1009.2425 [hep-lat], accepted for publication in *Phys. Rev.D*.